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by

G. Fusco and J. K. Hale

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Abstract

A scalar parabolic equation with nonconstant diffusion and nonlinear source term is considered and some aspects of the influence of changing the diffusion on existence, stability and bifurcation properties of the equilibria are discussed.

1. Introduction

We deal with existence stability and bifurcation properties of equilibria of the problem

$$(1)\qquad \begin{cases} u_t = (cu_x)_x + f(u), & x \in (-1,1) \\ u_x(-1,t) = u_x(1,t) = 0 \end{cases}$$

where $c > 0$ is a continuous function and f is C^1.

The initial value problem for (1) is well posed in the Sobolev space $H^1(-1,1)$, [1], and any bounded orbit approaches an equilibrium as $t \to \infty$ ([3], [5],[7]). Therefore a basic problem in understanding the dynamic of (1) is the description of the set of equilibria of (1) and of the way this set changes with the diffusion function c and with the source term f. Related important problems are the characterization of the pairs (c,f) such that (1) has stable nonconstant equilibria and to understand the role of bifurcation in the appearance of stable equilibria.

For any nonlinear function f, Chafee [6] proved that when c is constant, no stable nonconstant equilibrium exists. Chafee's result was generalized by Hale and Chipot [2] that showed that the same result holds true if $c \in C^2$ and $c_{xx} \le 0$. Finally, Yanagida [10] has shown that if c is written as $c = a^2$, $a > 0$ a necessary and sufficient condition for the nonexistence of a function f such that (1) has a stable nonconstant equilibrium is that $a_{xx} \le 0$. Other results concerning the existence of stable nonconstant equilibria are due to Matano ([7],[9]) that has shown that, if f is a cubic polynomial as $f = u - u^3$ and $c(x) \ge 1$ on intervals $[-1,\alpha],[\beta,1]$ and $\le \varepsilon$

on $[\gamma,\delta]$ $\alpha < \gamma < \delta < \beta$ and ε is sufficiently small, then (1) has a stable nonconstant equilibrium. Fife and Peletier [13] have also considered equations related to (1) which have stable nonconstant equilibria.

For the n dimensional version of problem (1) in a bounded domain Ω and with constant diffusion, Casten and Holland [11] and Matano [8] have shown that, if Ω is convex, any stable equilibrium must be a constant. Matano has also shown that, assuming f of the type $f = u - u^3$, for some nonconvex domains, there exist stable nonconstant equilibria. Hale and Vegas [4] have shown the existence of stable nonconstant equilibria for a large class of nonlinearities and for domains Ω_ε that can be considered as perturbations of a domain Ω_0 which is the union of two disjoint convex domains.

We assume that c is even and f is odd and such that

$$(2) \qquad \begin{cases} f(0) = f(1) = 0 \\ f(u) > 0 \quad \text{for} \quad u \in (0,1) \\ f(u) < 0 \quad \text{for} \quad u \in (1,\infty), \\ f'(0) \neq 0, \quad f'(1) \neq 0. \end{cases}$$

(see fig. 1.)

Under these assumptions we give an estimate of the number of equilibria of (1) in terms of c and f. We prove that for any f of type (2), if c is sufficiently close to the step function

$$(3) \qquad \tilde{c} = \begin{cases} 1 & \text{for} \quad x \in [-1,\ell] \cup [\ell,1], \\ c_0 > 0 & \text{for} \quad x \in (-\ell,\ell), \quad 0 < \ell < 1 \end{cases}$$

and c_0 is sufficiently small, problem (1) has at least a pair of stable nonconstant odd monotone equilibria. Finally, we show that if $c = c_\mu$ depends on

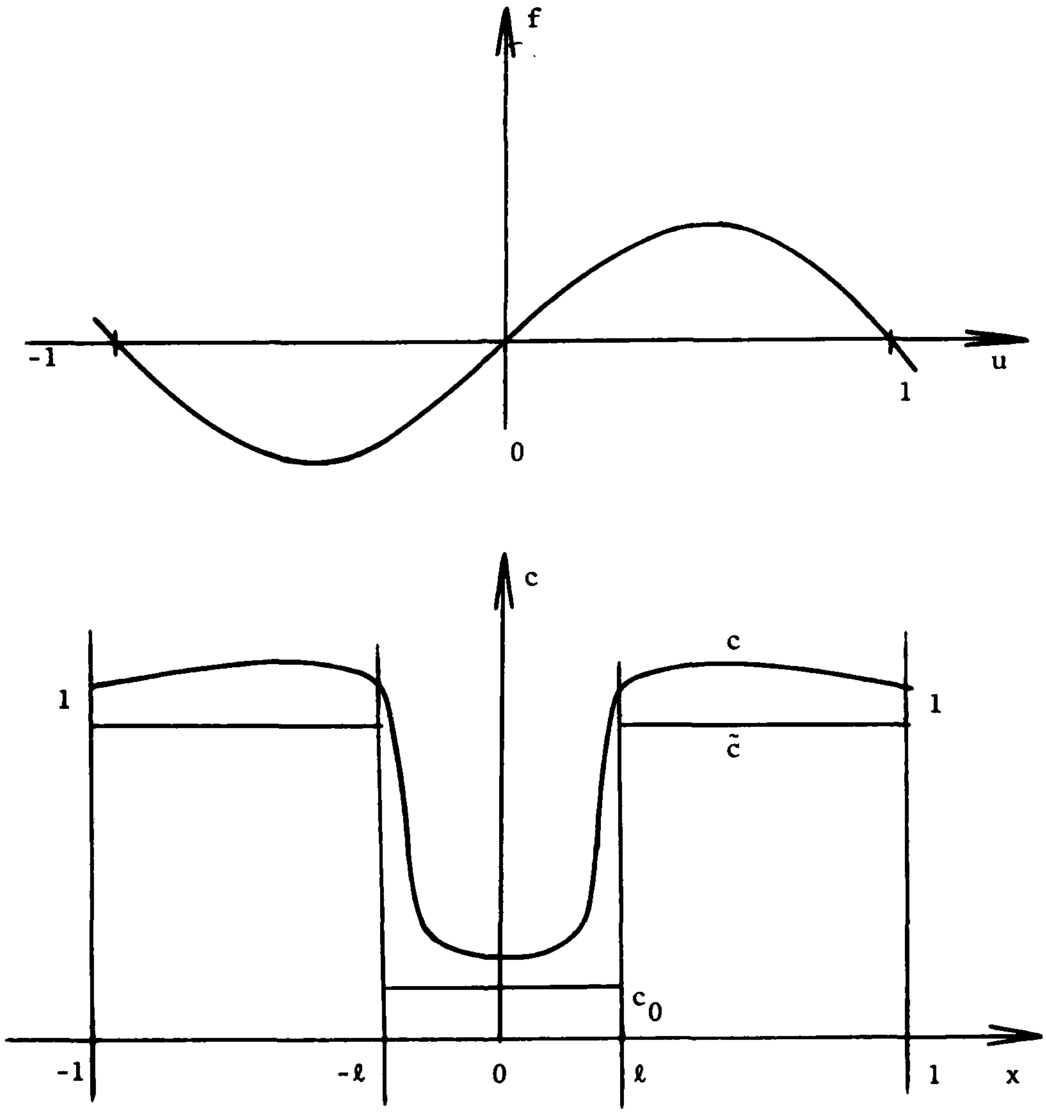

Fig. 1

a parameter $\mu \in [0,1]$ and u_μ is an equilibrium of (1) with exactly k zeros that bifurcates at $\mu = 0$ from the zero equilibrium and becomes stable at $\mu = 1$ then as μ goes from 0 to 1, u_μ must go through at least k secondary bifurcations.

2. Existence

We are interested in studying problem (1) for c in the set $\mathscr{C}$ of continuous and positive functions $c: [-1,1] \to \mathbb{R}$. Nevertheless, for the analysis that follows, in particular for the discussion of stability where we consider function c that are "close" to the step function $\tilde{c}$ defined by (3), it is convenient to study problem (1) for a wider class of diffusion functions c that are allowed to have jump discontinuities at a finite number of fixed points in $[-1,1]$. To keep the notation simple and since we suppose c even, we consider only the case of two points of discontinuity at $x = \pm\ell$, $0 < \ell < 1$. Everything we say extends to the case of any number of points of discontinuity.

Let $\tilde{\mathscr{C}}$ be the set of nonnegative even functions $c: [-1,1] \to \mathbb{R}$ which have continuous restrictions to $[0,\ell)$ and to $[\ell,1]$ and possess the left limit $c(\ell^-)$ of $c(x)$ as $x \to \ell$. For any $c \in \tilde{\mathscr{C}}$ let $J_c \subset \mathbb{R}^2$ be the set $J_c \overset{\text{def}}{=} \{(x,y) | x = \ell,\ y \in [c(\ell^-), c(\ell)]\}$ and $C \overset{\text{def}}{=} J_c \cup \text{graph } c$. We suppose that $\tilde{\mathscr{C}}$ is endowed with the topology associated with the following notion of convergence that allows a sequence of continuous functions to converge to a function that has a jump at $x = \pm\ell$: we say that $c_n \in \tilde{\mathscr{C}}$, $n = 1,\ldots,$ converges to $c \in \tilde{\mathscr{C}}$ if and only if C_n converges to C in the sense of Hausdorff as $n \to \infty$.

The class of diffusion function that we are going to consider is the subset $\tilde{\mathscr{C}}^+ \subset \tilde{\mathscr{C}}$ defined by the condition $\inf c > 0$. Clearly, $\mathscr{C}$ is a dense subset of $\tilde{\mathscr{C}}^+$ and if we assume in $\mathscr{C}$ the topology of uniform convergence in $[-1,1]$, then $\mathscr{C}$ is continuously embedded in $\tilde{\mathscr{C}}^+$. Henceforth we allow c in

problem (1) to be a generic $c \in \tilde{\mathscr{L}}^+$. This requires that (1) be complemented with the jump conditions

$$c(\pm\ell^+)u_x(\pm\ell^+) = c(\pm\ell^-)u_x(\pm\ell^-),$$

therefore the equilibrium problem corresponding to (1) becomes

$$\text{(4)} \qquad \begin{cases} (cu_x)_x + f(u) = 0 \\ u_x(-1) = u_x(1) = 0 \\ c(\pm\ell^+)u_x(\pm\ell^+) = c(\pm\ell^-)u_x(\pm\ell^-), \end{cases}$$

and reduces to the standard problem for $c \in \mathscr{L}$.

By letting $u = u$, $v = cu_x$, problem (4) transforms in the equivalent system

$$\text{(5)} \qquad \begin{cases} u_x = \frac{v}{c}, \\ v_x = -f(u), \\ v(-1) = v(1) = 0. \end{cases}$$

Note that the jump conditions express just continuity of v at $x = \pm\ell$ and therefore they are included in the requirement that u,v be continuous in $[-1,1]$.

The hypothesis on f and a maximum principle argument imply that solutions of (4) or (5) satisfy $-1 \leq u(x) \leq 1$, therefore we can also assume that f is bounded so that the solution $u(c,a,x)$, $v(c,a,x)$ of the initial value problem

$$(6)\qquad \begin{cases} u_x = \dfrac{v}{c}, \\ v_x = -f(u), \\ u(-1) = a,\ v(-1) = 0 \end{cases}$$

is defined for all $(c,a,x) \in \tilde{\mathscr{L}}^+ \times [-1,1] \times [-1,1]$.

Lemma 1. $u(c,a,x)$, $v(c,a,x)$ *are continuous functions of* (c,a) *uniformly in* x *and possess a continuous first derivative with respect to* a *and also with respect to* x *except possibly at* $x = \pm\ell$.

The proof of this lemma is a standard application of the general theory of differential equations.

To discuss the existence of space dependent equilibria of (1), i.e., the existence of nonconstant solutions of (4), we note that these solutions are in one to one correspondence with the $a \neq -1,0,1$ such that $v(c,a,1) = 0$. If, for $a \neq 0$, we let $\delta(c,a,x)$ be the angle (positive clockwise around x in space u,v,x) swept by the vector $\vec{u}(c,a,x')$ defined by

$$\vec{u}(c,a,x') = \begin{bmatrix} u(c,a,x') \\ v(c,a,x') \\ 0 \end{bmatrix},$$

when x' goes from -1 to x, then a necessary and sufficient condition in order that $v(c,a,1) = 0$ for some $a \neq -1,0,1$ is that $\delta(c,a,1)$ be equal to πk for some integer $k \neq 0$.

The angle $\delta(c,a,x)$ can be defined also for $a = 0$ so that $\delta(c,a,x)$ is continuous in (c,a,x). In fact by performing the polar coordinate trans-

formation $u = \rho\cos\delta$, $v = -\rho\sin\delta$ it is found that $\delta(c,a,\cdot)$ is the solution of the problem

$$(7)\qquad \begin{cases} \delta_x = \dfrac{1}{c(x)}\sin^2\delta + \dfrac{f(u(c,a,x))}{u(c,a,x)}\cos^2\delta, \\ \delta(-1) = 0. \end{cases}$$

Moreover, since by lemma 1, $u(c,a,x)$ is continuous in (c,a) uniformly in x and $u(c,o,x) = 0$, it follows that if (c',a) converges to (c,o) in $\tilde{\mathscr{C}}^+ \times [-1,1]$, $f(u(c',a,x))/u(c',a,x)$ converges uniformly to $f'(0)$ in $[-1,1]$. This implies that as $(c',a) \to (c,0)$, $\delta(c',a,x)$ converges uniformly to the solution $\delta(c,0,\cdot)$ of the problem

$$(8)\qquad \begin{cases} \delta_x = \dfrac{1}{c(x)}\sin^2\delta + f'(0)\cos^2\delta, \\ \delta(-1) = 0. \end{cases}$$

From (7),(8) and lemma 1 it also follows that $\delta(c,a,x)$ is continuously differentiable with respect to a. We also note that $\delta(c,\pm1,x) = 0$ and that, for $a \in (1,1)$, $\delta(c,a,x)$ is an increasing function of x because the right hand sides of (7),(8) are > 0.

For later use we also introduce the angle $\sigma(c,a,x)$ which is defined as $\delta(a,x)$ with the vector $\vec{u}(c,a,x)$ replaced by its derivative $\vec{u}_a(c,a,x)$ with respect to a. It may be useful to note that, if Σ is the surface in the space of u,v,x defined by the solutions of (6), then $\vec{u}_a(c,a,x)$ is tangent to the cross section of Σ at x at the point $(u(c,a,x), v(c,a,x),x)$ (see fig. 2).

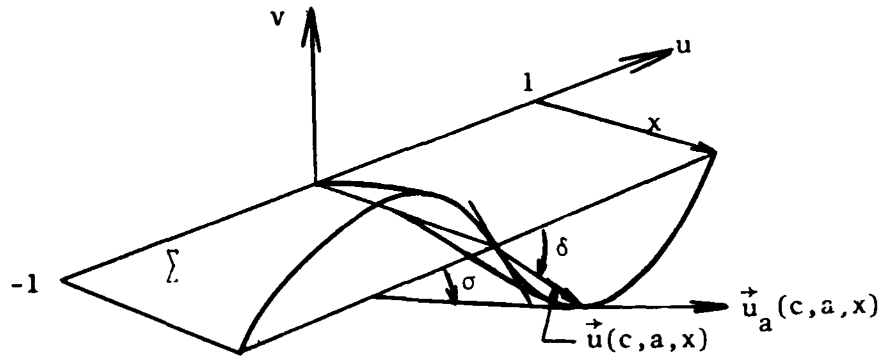

Fig. 2

It is easily seen that $\sigma(c,a,\cdot)$ is the solution of the problem

$$(9)\qquad \begin{cases} \sigma_x = \dfrac{1}{c(x)} \sin^2\sigma + f'(u(c,a,x))\cos^2\sigma, \\ \sigma(-1) = 0. \end{cases}$$

Now consider an interval $[-\bar{\ell},\bar{\ell}]\subset[-1,1]$ and let $\bar{c}$ be the supremum of $c(x)$ in $[-\bar{\ell},\bar{\ell}]$. Then for $x \in [-\bar{\ell},\bar{\ell}]$ (8) implies

$$(10)\qquad \delta_x \geq \frac{1}{\bar{c}} \sin^2\delta + f'(0)\cos^2\delta.$$

Since a simple computation shows that the solutions of (10) with the equality sign increase of π each time that x increases of $\pi(f'(0)/\bar{c})^{-1/2}$, from (10) and the fact that $\delta(c,a,x)$ is a nondecreasing function of x, it follows that

$$\delta(c,a,1) \geq \pi\text{. Integer part of } \left[\frac{2\bar{\ell}}{\pi}\left(\frac{f'(0)}{\bar{c}}\right)^{1/2}\right] .$$

This estimate together with the continuity of $\delta(c,\cdot,1)$ and the fact that

$\delta(c,\pm 1,1) = 0$ imply

<u>Theorem 1.</u> <u>The number</u> N <u>of nonconstant equilibria of</u> (1) <u>satisfies the condition</u>

$$N \geq 2. \text{ Integer part of } \left[\frac{2\bar{\ell}}{\pi}\left(\frac{f'(0)}{\bar{c}}\right)^{1/2}\right]. \tag{11}$$

<u>Remark.</u> In the proof of theorem 1. no use was made of the eveness of c and oddness of f. Thus theorem 1. holds for generic c,f. We also note that the conclusion of theorem 1. is also true if $[-\bar{\ell},\bar{\ell}]$ is replaced by a measurable set $E \subset [-1,1]$ of measure $2\bar{\ell}$.

Let $s_k = \{a | \delta(c,a,1) = k\pi\}$. The set s_k can be identified with the set of equilibria of (1) that have exactly k zeros. If the right hand side of (12) is $\geq 2k$, then s_k is nonempty and by mean of equation (7) it is possible to obtain some information on the "shape" of equilibria. To this aim let $0 < \bar{u} < 1$ and $a \in s_k$ be given and $\mathcal{J} \subset [-1,1]$ be the set where $|u(a,x)| < \bar{u}$. To get a bound for the measure of $\mathcal{J} \cap (-\bar{\ell},\bar{\ell})$ we let (x_1,x_ℓ) be the smallest interval containing $\mathcal{J} \cap (-\bar{\ell}\ \bar{\ell})$, $L = x_2 - x_1$ its length, and $\bar{\eta} \overset{\text{def}}{=} \min\limits_{|u| \leq \bar{u}} f(u)/u$, then by applying to (7) the same procedure used for deriving (10) from (8) we obtain

$$\delta_x \geq \frac{1}{\bar{c}} \sin^2\delta + \bar{\eta} \cos^2\delta \quad , \quad x \in \mathcal{J} \cap (-\bar{\ell},\bar{\ell})$$

and therefore, since $\delta(c,a,x)$ is an increasing function of x,

$$L \leq \int_{\delta(c,a,x_1)}^{\delta(c,a,x_2)} \frac{d\delta}{\frac{1}{\bar{c}} \sin^2\delta + \bar{\eta}\cos^2\delta} .$$

From this estimate it follows

$$L < k\pi\left(\frac{\bar{c}}{\bar{\eta}}\right)^{1/2} \tag{12}$$

because $\delta(c,a,x_2) - \delta(c,a,x_1) < k\pi$. The estimate (12) shows that $\inf_{a\in s_k}\{|u(c,a,x)|\}$ converges in measure to 1 in $(-\bar{\ell},\bar{\ell})$ as $\bar{c} \to 0$.

For $k \neq 1$ nothing can be said on the behavior of solutions in s_k outside the interval $(-\bar{\ell},\bar{\ell})$ and solutions in s_k could be almost trivial in the sense that they could be near zero outside $(-\bar{\ell},\bar{\ell})$ and oscillate in $(-\bar{\ell},\bar{\ell})$. This can not happen when $k = 1$ because solutions in s_1 are monotone and therefore if there is a point $\bar{x} \in (-\bar{\ell},\bar{\ell})$ where $|u(c,a,\bar{x})|$ is near 1 the same is true in $[-1,\bar{x}]$ or in $[\bar{x},1]$.

In what follows we are interested in solutions of (4) that are odd functions of x. It is easily seen that, due to the assumption that c is even and f is odd, when on the basis of (11) it is possible to conclude that s_1 is nonempty, then it also contains at least a pair of odd solutions that transform in each other under the transformation $x \to -x$. Clearly, if $u(c,a,\cdot)$ is one of these odd solutions, and $|u(c,a,x)\mp 1| < \varepsilon$ in $[-1,\bar{x}]$ then $|u(c,a,x)\pm 1| < \varepsilon$ in $[-\bar{x},1]$. Therefore on the basis of (12) we have

<u>Theorem 2.</u> <u>For any</u> c <u>such that the right hand side of</u> (11) <u>is</u> ≥ 2, <u>problem</u> (1) <u>has an equilibrium which is an odd and increasing function of</u> x. <u>If</u> c <u>is deformed so that</u> $\bar{c} \to 0$, <u>then all odd increasing equilibria of</u> (1) <u>converge to the function</u>

$$z = \begin{cases} -1 & \text{for} \quad x \in [-1,0) \\ 0 & \text{for} \quad x = 0 \\ 1 & \text{for} \quad x \in (0,1] \end{cases}$$

and the convergence is uniform in compact sets in $[-1,0) \cup (0,1]$.

In the statement of theorem 2. and in the following, we always refer to the increasing equilibrium, with it being understood that there is also a decreasing equilibrium that transforms into the other one under the transformation $x \to -x$.

3. Stability.

Let $\tilde{c}^0 \in \tilde{\mathscr{C}}$ be the function defined by

$$\tilde{c}^0 = \begin{cases} 1 & x \in [-1,-\ell) \cup (\ell,1], \\ 0 & x \in (-\ell,\ell). \end{cases}$$

In this section we prove the following

Theorem 3. Let f be a continuously differentiable odd function that satisfies (2). Then there is a set $W \subset \mathscr{C}$ such that

(i) W is open and connected in $\mathscr{C}$

(ii) $\tilde{c}^0$ belongs to the closure of W in $\tilde{\mathscr{C}}$

(iii) for any $c \in W$ problem (1) has an odd increasing (and an odd decreasing) equilibrium which is stable.

Note that theorem 3. implies

Corollary. For any odd C^1-function f that satisfies (2) there is a $c \in \mathscr{C}$ such that problem (1) has a stable nonconstant equilibrium.

To prove theorem 3. we need a few lemmas.

Lemma 2. *If* $u(c,a,\cdot)$ *is an equilibrium of* (1) *and* λ *is the first eigenvalue of the linear problem*

$$(13)\qquad \begin{cases} (cw_x)_x + f'(u(c,a,x))w = \lambda w, \\ w_x(-1) = w_x(1) = 0 \\ c(\pm\ell^+)w_x(\pm\ell^+) = c(\pm\ell^-)w_x(\pm\ell^+), \end{cases}$$

then $u(c,a,\cdot)$ *is stable if* $\lambda < 0$, *unstable if* $\lambda > 0$.

The proof of this lemma is given in [1].

Lemma 3. *Let* $\tilde{\mathscr{S}} \subset \tilde{\mathscr{C}}^+$ *be the set of functions* c *such that* (1) *has a stable nonconstant equilibrium the stability of which can be ascertained by the fact that the largest eigenvalue of the linear problem* (13) *is negative. Then* $\tilde{\mathscr{S}}$ *is open in* $\tilde{\mathscr{C}}^+$.

Proof. If $\tilde{\mathscr{S}}$ is empty, the lemma is obvious. Therefore we assume that $\tilde{\mathscr{S}}$ is nonempty. Then there exist $c \in \tilde{\mathscr{C}}$, $a \in (-1,1) \setminus \{0\}$ and $k > 0$ such that

$$\delta(c,a,1) = k\pi,$$

and the largest eigenvalue λ of problem (13) is negative. If one lets $w = r\cos\nu$, $cw_x = -r\sin\nu$ in equation (13), it is found that r,ν satisfy

$$(14)\qquad r_x = \sin\nu\cos\nu\left(f'(u(c,a,x)) - \frac{1}{c(x)} - \lambda\right)r,$$

$$(15)\qquad \nu_x = \frac{1}{c(x)}\sin^2\nu + (f'(u(c,a,x))-\lambda)\cos^2\nu,$$

with the boundary conditions $\nu(-1) = 0$, $\nu(1) = i\pi$ for some integer i.

But i must be zero because the eigenfunction w corresponding to the largest eigenvalue never vanishes and therefore ν must stay in the interval $(-\pi/2,\pi/2)$. Since λ is negative, for $\nu = \sigma$, the right hand side of (9) is always smaller than the right hand side of (15); therefore, we have $\sigma(c,a,1) < \nu(1) = 0$. On the other hand, equation (9) implies $\sigma(c,a,x) > -\frac{\pi}{2}$. It follows that $\sin(\delta(c,a,1) - \sigma(c,a,1)) \neq 0$. Since the derivative of δ with respect to a is related to σ by

$$\frac{\rho^2\delta_a^2}{\rho^2\delta_a^2+\rho_a^2} = \sin^2(\delta-\sigma) \tag{16}$$

it results $\delta_a(c,a,1) \neq 0$. Then the lemma follows by the implicit function theorem.

<u>Remark.</u> Lemma 3 is actually a special case of a general situation. In fact, the largest eigenvalue being negative for an equilibrium point u_0, implies the semigroup generated by the linear variational equation is exponentially asymptotically stable. Thus, small perturbation in c will yield another equilibrium point near u_0 also has the largest eigenvalue negative and it will be stable.

In the proof of lemma 3, we have seen that $\sigma(c,a,1) < 0$ is a necessary condition for λ to be negative. We note that this condition is also sufficient. This follows from the fact the solution of (15) depends continuously on λ, coincides with $\sigma(c,a,\cdot)$ for $\lambda = 0$, and increases unboundedly as $\lambda \to -\infty$ for $x \neq -1$. Therefore, if $\sigma(c,a,1)$ is negative, there exists a unique negative λ_0 such that the solution of (15) vanishes at 1. Then, if $r(\cdot)$ is any non zero solution of (14), $w(\cdot) = r(\cdot)\cos\nu(\cdot)$ is an eigen-

function of (13) that does not vanish in [-1,1]. Thus $\lambda_0 < 0$ is the largest eigenvalue of (13). Therefore, we can state

<u>Proposition 1.</u> <u>A necessary and sufficient condition in order that the largest eigenvalue of problem</u> (13) <u>be negative is that</u> $\sigma(c,a,1)$ <u>be negative.</u>

<u>Lemma 4.</u> <u>Let</u> $\tilde{c} \in \tilde{\mathcal{S}}^+$ <u>be a function of type</u> (3) <u>and</u> $(1)_{\tilde{c}}$ <u>be problem</u> (1) <u>with</u> $c = \tilde{c}$. <u>Then if</u> c_0 <u>is sufficiently small, problem</u> $(1)_{\tilde{c}}$ <u>has an odd increasing equilibrium</u> $u(\tilde{c},a,\cdot)$ <u>such that the largest eigenvalue of the corresponding linear problem</u> $(13)_{\tilde{c}}$ <u>is negative.</u>

<u>Proof.</u> By theorem 1, if $c_0 < \frac{4\ell^2}{\pi^2} f'(0)$ there exists an $\tilde{a} \in (-1,1) \setminus \{0\}$ such that $u(\tilde{c},\tilde{a},\cdot)$ is an increasing equilibrium of $(1)_{\tilde{c}}$. The same condition together with the evenness of c and the oddness of f ensure that $\tilde{a}$ can also be chosen so that $u(\tilde{c},\tilde{a},\cdot)$ is an odd function. To prove that the largest eigenvalue of the linearized problem at $u(\tilde{c},\tilde{a},\cdot)$ is negative if c_0 is sufficiently small, we recall [12] that the eigenvalues of (13) do not decrease if $f'(u(c,a,x))$ is replaced by a function $q(x) \geq f'(u(c,a,x))$. It follows that, if we let $\bar{q} = \max\limits_{u \geq u(\tilde{c},\tilde{a},\ell)} f'(u(\tilde{c},\tilde{a},\ell))$, it suffices to show that for c_0 small, the largest eigenvalue of

$$(17) \qquad \begin{cases} w_{xx} + \bar{q}w = \lambda w \quad , & x \in (-1,-\ell) \cup (\ell,1) \\ c_0 w_{xx} + f'(u(\tilde{c},\tilde{a},x))w = \lambda w, & x \in (-\ell,\ell) \end{cases}$$

$$(18) \qquad \begin{cases} w_x(-1) = w_x(1) = 0 \\ c_0 w_x(-\ell^+) = w_x(-\ell^-), \\ w_x(\ell^+) = c_0 w_x(\ell^-) \end{cases}$$

is negative.

From a result of Yanagida [10], it follows that the largest eigenvalue of this problem is negative if there is a strictly positive function w_0 that makes the left hand sides of (17) equal to zero, satisfies the last two equations (18) and moreover, is such that

(19) $$w_{ox}(-1) < 0, \quad w_{0x}(1) > 0.$$

We look for an even such w_0 and therefore, we assume $w_{ox}(0) = 0$ and consider only the interval $[0,1]$. Since $f'(1) < 0$ and by theorem 2, $u(\tilde{c},\tilde{a},\ell) \to 1$ as $c_0 \to 0$, $\bar{q}$ is negative for small value of c_0. Therefore, if w_0 exists, in the interval $[\ell,1]$, it must have the expression

(20) $$w_0(x) = A \sinh [(-\bar{q})^{1/2}(x-\ell)] + B \cosh [(-\bar{q})^{1/2}(x-\ell)],$$

and the coefficients A,B must satisfy the conditions

(21) $$A > 0, \frac{B}{A} > - \tanh [(-\bar{q})^{1/2}(1-\ell)]$$

ensuring that $w_0(x)$ is positive in $[\ell,1]$ and $w_{ox}(1) > 0$.

To compute $w_0(x)$ in the interval $[0,\ell]$, we must solve the problem

(22) $$\begin{cases} c_0 w_{oxx} + f'(u(\tilde{c},\tilde{a},x))w_0 = 0, & x \in (0,\ell), \\ w_0(0) = C, \quad w_{0x}(0) = 0, \end{cases}$$

where C is a positive constant to be chosen later.

From now on we set for simplicity $\tilde{u} = u(\tilde{c},\tilde{a},\cdot)$, $\bar{u} = \tilde{u}(\ell)$, $\bar{\bar{u}} = \tilde{u}(1)$. In order to solve this problem we must overcome the difficulty lying in the fact that u is only known to be an odd increasing solution of problem $(4)_{\tilde{c}}$. To this

end we observe that since $\tilde{u}$ is increasing, we can perform the change of variable $x = \tilde{u}^{-1}(u) \overset{\text{def}}{=} \xi(u)$. By making this change of variable in $(4)_{\tilde{c}}$ and by observing that the oddness of $\tilde{u}$ implies $\xi(0) = 0$, we see that ξ satisfies

$$(23)\qquad \begin{cases} -c_0 \dfrac{\xi''}{\xi'^3} + f(u) = 0 & u \in (0,\bar{u}), \\ -\dfrac{\xi''}{\xi'^3} + f(u) = 0 & u \in (0,\bar{\bar{u}}), \end{cases}$$

$$(24)\qquad \begin{cases} \xi(0) = 0 \quad , & \xi(\bar{u}) = \ell, \\ \lim\limits_{u\to\bar{\bar{u}}} \xi(u) = 1, & \lim\limits_{u\to\bar{\bar{u}}} \xi'(u) = \infty. \end{cases}$$

By using the fact that $\dfrac{d}{dx} = \dfrac{1}{\xi'}\dfrac{d}{du}$, one sees that the same change of variables applied to (22) yields

$$(25)\qquad \begin{cases} \dfrac{c_0}{\xi'^2} w_0 - c_0\dfrac{\xi''}{\xi'^3} w_0' + f'w = 0, & u \in (0,\bar{u}), \\ w_0(0) = C, \quad w_0'(0) = 0, \end{cases}$$

where w_0 has been identified with the function $w_0(\xi(\cdot))$. From (23) it follows that, for $u \in (0,\bar{u})$

$$\frac{c_0}{\xi'^2} = 2\int_u^1 f(s)ds + K \overset{\text{def}}{=} g(u),$$

where $K > -2\int_{\bar{u}}^1 f(s)ds$ is an integration constant that together with $\bar{u},\bar{\bar{u}}$ satisfies the conditions

$$
(26)\qquad \begin{cases} \displaystyle\int_{\bar{u}}^{\bar{\bar{u}}} \frac{du}{\left(2\int_u^{\bar{\bar{u}}} f(s)ds\right)^{1/2}} = 1 - \ell, \\[2ex] \displaystyle c_0 g(\bar{u}) = 2\int_{\bar{u}}^{\bar{\bar{u}}} f(s)ds, \end{cases}
$$

which correspond to the last two conditions (24). Sinc $g' = -2f$ and the first equation (23) implies that the coefficient of w_0' in equation (25) is equal to $-f$, equation (25) becomes

$$
gw_0'' - fw_0' + f'w_0 = (gw_0' + fw_0)' = 0.
$$

Thus $gw_0' + fw_0 = \text{const} = 0$ because $w_0'(0) = 0$ and $f(0) = 0$.

It follows that, with a proper choice of the constant C appearing in (25),

$$
(27)\qquad w_0 = g^{1/2}, \qquad (\text{for } u \in [0,\bar{u}]).
$$

If the expressions (20)(27) are patched together at $x = \ell$ (corresponding to $u = \bar{u}$) by imposing the conditions

$$
w_0(\ell^+) = w_0(\ell^-), \quad w_{0x}(\ell^+) = c_0 w_{0x}(\ell^-),
$$

it is found that

$$
A = (g(\bar{u}))^{1/2}, \quad B = -c_0^{1/2}\frac{f(\bar{u})}{(-\bar{q})^{1/2}}.
$$

Therefore, it follows that if

$$
(28)\qquad \frac{c_0^{1/2} f(\bar{u})}{(-\bar{q})^{1/2}(g(\bar{u}))^{1/2}} < \tanh\left[(-\bar{q})^{1/2}(1-\ell)\right],
$$

then w_0 satisfies all the conditions ensuring that the largest eigenvalue of problem (17)(18) is negative. We shall prove that this is the case for c_0 sufficiently small. The proof is a discussion of the asymptotic dependence of $\bar{u}, g(\bar{u})$ on c_0 defined by equations (26) for $c_0 \to 0$.

By the change of variables $u = \bar{u} + (\bar{\bar{u}}-\bar{u})\tau$, $s = u + (\bar{\bar{u}}-u)\sigma$, the first of equations (26) transforms as

$$(29) \qquad \frac{1}{2^{1/2}} \int_0^1 (1-\tau)^{-1/2} \left(\int_0^1 \frac{f(1+\bar{\delta}(\tau,\sigma))}{\bar{\delta}(\tau,\sigma)} (\tau-\bar{\alpha}+(1-\tau)\sigma)d\sigma \right)^{-1/2} d\tau = 1 - \ell,$$

where

$$\bar{\delta}(\tau,\sigma) = -[(1-\bar{\bar{u}}) + (\bar{\bar{u}}-\bar{u})(1-\tau)(1-\sigma)],$$

$$\bar{\alpha} = \frac{1-\bar{u}}{\bar{\bar{u}}-\bar{u}} > 1.$$

As $\bar{u}, \bar{\bar{u}}$, also $\bar{\delta}$ and $\bar{\alpha}$ depend on c_0. Let $\alpha = \limsup_{c_0 \to 0} \bar{\alpha}$. Since $\bar{u} \to 1$, $\bar{\bar{u}} \to 1$ as $c_0 \to 0$, the above expression of $\bar{\delta}(\tau,\sigma)$ implies that $\bar{\delta}(\tau,\sigma) \to 0$ uniformly as $c_0 \to 0$. Therefore, $f(1) = 0$ implies that the ratio $f(1+\bar{\delta}(\tau,\sigma))/\bar{\delta}(\tau,\sigma)$ converges uniformly to $f'(1) \neq 0$. This and equation (29) imply $\alpha < \infty$.

We also have

$$(30) \qquad \liminf_{c_0 \to 0} \frac{1}{(1-\bar{u})^2} \int_{\bar{u}}^{\bar{\bar{u}}} f(s)ds = \liminf_{c_0 \to 0} \frac{1}{\bar{\alpha}^2} \int_{\bar{u}}^{\bar{\bar{u}}} \frac{f(s)ds}{(\bar{\bar{u}}-\bar{u})^2} =$$

$$\liminf_{c_0 \to 0} \frac{1}{\bar{\alpha}^2} \int_0^1 \frac{f(1+\bar{\delta}'(\tau))}{\bar{\delta}'(\tau)} (\tau-\bar{\alpha})d\tau,$$

with $\bar{\delta}'(\tau) = (\bar{u}-1) + (\bar{\bar{u}}-\bar{u})\tau$. Since $\bar{\delta}'(\tau) \to 0$ uniformly as $c_0 \to 0$, the ratio $f(1+\bar{\delta}'(\tau))/\bar{\delta}'(\tau) \to f'(1) < 0$ uniformly as $c_0 \to 0$. From this and the fact that $\bar{\alpha}$ is > 1, it follows

(31) $$\liminf_{c_0 \to 0} \frac{1}{\bar{\alpha}^2} \int_0^1 \frac{f(1+\bar{\delta}'(\tau))}{\bar{\delta}'(\tau)} (\tau-\bar{\alpha})d\tau \geq -f'(1)\int_0^1 (1-\tau)d\tau = -\frac{1}{2}f'(1).$$

Equations (30)(31) and the second condition (26) imply

(32) $$\liminf_{c_0 \to 0} \frac{c_0 g(\bar{u})}{(1-\bar{u})^2} \geq -f'(1).$$

Therefore, by taking into account that $\lim_{c_0 \to 0} \bar{q} = f'(1)$, we obtain

$$\limsup_{c_0 \to 0} \frac{c_0^{1/2} f(\bar{u})}{(-\bar{q})^{1/2}(g(\bar{u}))^{1/2}} = \frac{1}{(-f'(1))^{1/2}} \limsup_{c_0 \to 0} c_0 \frac{f(1+(\bar{u}-1))}{(1-\bar{u})} \frac{(1-\bar{u})}{c_0^{1/2}(g(\bar{u}))^{1/2}} = 0,$$

that is: the left hand side of (28) converges to zero as $c_0 \to 0$. This proves the lemma because the limit of the right hand side is > 0.

<u>Proof of theorem 3.</u> By lemma 4, there is a number $\varepsilon > 0$ such that, if $\tilde{\gamma} \subset \tilde{\mathscr{L}}^+$ is the curve $\tilde{\gamma} \overset{\text{def.}}{=} \{\tilde{c} | 0 < c_0 < \varepsilon\}$ and $\tilde{c} \in \tilde{\gamma}$, then problem $(1)_{\tilde{c}}$ has an odd increasing equilibrium which is stable. Since by lemma 3 $\tilde{\mathscr{S}}$ is open in $\tilde{\mathscr{L}}^+$ there exists an open neighborhood $\tilde{W}$ of $\tilde{\gamma}$ in $\tilde{\mathscr{L}}^+$ such that for $c \in \tilde{W}$ problem (1) has a stable equilibrium u_c. It is easy to see that $\tilde{W}$ can be chosen so that u_c is odd and increasing. In fact from the proof of lemma 3 it follows that for c in a neighborhood of $\tilde{c} \in \tilde{\gamma}$, u_c is the only equilibrium in a neighborhood of $u_{\tilde{c}}$. On the other hand the evenness

of c and the oddness of f imply that also v_c defined by $v_c(x) = -u_c(-x)$ is an equilibrium of (1). Since $u_{\tilde{c}}$ is odd and $u_c \to u_{\tilde{c}}$ as $c \to \tilde{c}$ also v_c converges to $u_{\tilde{c}}$ as $c \to \tilde{c}$. This contradicts uniqueness of u_c unless u_c is odd and therefore proves oddness. Since u_c is close to u_c it vanishes only at $x = 0$. From this and the fact that solutions of (4) with only one zero are monotone, it follows that u_c is increasing. The mapping $c_0 \to \tilde{c}$ is continuous as a map from $(0,\varepsilon)$ into $\tilde{\mathscr{C}}^+$. Therefore γ is locally compact as a subset of $\tilde{\mathscr{C}}^+$. Thus, by standard arguments there is a continuous function $\phi : (0,1) \to \mathscr{C}$ such that the curve $\gamma = \{c \mid c = \phi(s),\ s \in (0,1)\}$ is contained in $\tilde{W}$ and $\tilde{c}^0$ is in the closure of γ in $\tilde{\mathscr{C}}$. Since $\tilde{W}$ is open and $\mathscr{C}$ is continuously embedded in $\tilde{\mathscr{C}}^+$, $\tilde{W} \cap \mathscr{C}$ is open in $\mathscr{C}$. From this and the continuity of ϕ it follows that there is a subset $W \subset \tilde{W} \cap \mathscr{C}$ which is open and connected in $\mathscr{C}$ and contains γ. Since $\tilde{c}^0$ is in the closure of γ in $\tilde{\mathscr{C}}$ the proof is completed.

4. Secondary bifurcation

In this section we consider a family $(c_\mu)_{\mu\in[-1,1]} \subset \mathscr{C}$ of diffusion functions c_μ depending continuously on a parameter μ. We let $\delta(\mu,a,x) \overset{\text{def}}{=} \delta(c_\mu,a,x)$ and assume that $c_{\mu_2} < c_{\mu_1}$ for $\mu_2 > \mu_1$ and that $\delta(0,0,1) = k\pi$ for some $k > 0$. Then equation (8) implies that $\delta(\mu,0,1) < k\pi$ for $\mu < 0$ and $\delta(\mu,0,1) > k\pi$. If we also assume that f satisfies the condition

$$f(u) < f'(0)u \quad , \quad u \in (0,1], \tag{33}$$

then from (7)(8) it follows $\delta(\mu,a,1) < \delta(\mu,0,1)$ for $a \neq 0$, $\mu \in [-1,1]$. Therefore s_k is empty for $\mu \leq 0$, nonempty for $\mu > 0$. Thus $\mu = 0$ is a

bifurcation point. It is easy to see that in this situation, for $\mu > 0$ and small, s_k contains solutions that are small and converge to zero as $\mu \to 0$, i.e., solutions that bifurcate from the zero solution. These solutions are unstable for μ small because the largest eigenvalue of problem (13)with $u \equiv 0$ is $f'(0) > 0$ and the eigenvalues of (13) are continuous functions of $c \in \mathscr{L}$. On the other hand we have seen in theorem 3 that if c is suitably chosen then there exist stable nonconstant equilibria of (1). Therefore, it can be expected that, if u_μ is a continuous function of $\mu \in [0,1]$ such that $u_0 = 0$, u_μ is a solution of (4) in s_k for $\mu \in (0,1]$, and u_1 is stable, some kind of secondary bifurcation takes place at some $\mu \in (0,1)$. This conjecture is true, we have in fact the following.

Theorem 4. Suppose that c_μ is as before, f satisfies (33), u_s is an equilibrium of $(1)_{c_\mu}$ which is equal to zero for $\mu = 0$, has exactly k zeros for $\mu \in (0,1]$, depends continuously on μ and u_1 is stable (in the sense that the largest eigenvalue of the linearized problem at u_1 is negative), then there exist numbers $0 < \mu_1 < \ldots < \mu_k < 1$ such that μ_i, $i = 1,\ldots,k$ is a bifurcation point.

Proof. Let $a_\mu \overset{\text{def}}{=} u_\mu(-1)$. Then $a_0 = 0$ and therefore (7),(8) imply $\sigma(0,a_0,1) = \delta(0,a_0,1)$. We also have $\delta(\mu,a_\mu,1) = k\pi$ for $\mu \in (0,1]$, thus by the continuity of δ with respect to c,a and the continuity of c_μ, u_μ with respect to μ, it follows $\sigma(0,a_0,1) = k\pi$. On the other hand Proposition 1 and the stability of u_1 imply $\sigma(1,a_1,1) < 0$. Therefore, by continuity there exist $0 < \mu_1 < \ldots < \mu_k < 1$ such that

$$\sigma(\mu_i, a_{\mu_i}, 1) = (k-i)\pi \qquad i = 1,\ldots,k.$$

Moreover, it is obvious that μ_i, $i = 1,\ldots,k$ can be chosen so that in any neighborhood of μ_i there exist $\bar{\mu} < \mu_i < \bar{\bar{\mu}}$ such that $\sigma(\bar{\mu}, a_{\bar{\mu}}, 1) \geq (k-i)\pi > \sigma(\bar{\bar{\mu}}, a_{\bar{\bar{\mu}}}, 1)$. This on the basis of the geometrical meaning of the angle σ implies that μ_i is a bifurcation point.

Theorem 4 says that going through k secondary bifurcations is a necessary condition in order that an equilibrium with k zeros that bifurcates from the zero solution becomes stable. From the proof of the theorem and Proposition 1 it follows that if, as μ goes from 0 to 1, u_μ experiences exactly k bifurcations at $0 < \mu_1 < \ldots < \mu_k < 1$ each one of which is simple in the sense that at any μ_i two new solutions bifurcating from u_{μ_i} appear, then u_1 is stable (see fig. 3 for the case $k = 2$). This observation shows that in a certain sense the converse of theorem 4 is also true.

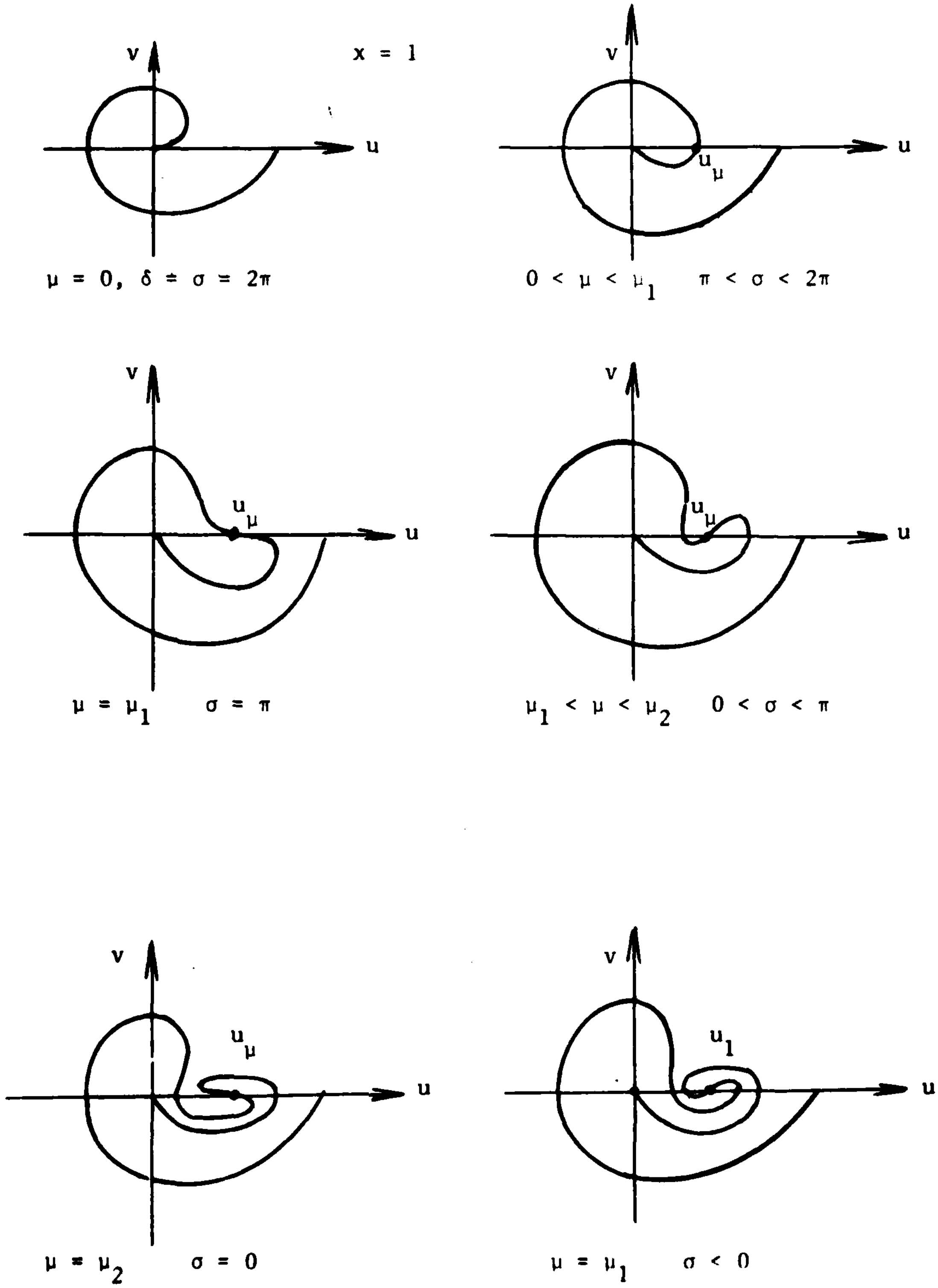
x = 1
v
u
u_μ
$\mu = 0,\ \delta \doteq \sigma = 2\pi$
$0 < \mu < \mu_1 \quad \pi < \sigma < 2\pi$
$\mu = \mu_1 \quad \sigma = \pi$
$\mu_1 < \mu < \mu_2 \quad 0 < \sigma < \pi$
$\mu = \mu_2 \quad \sigma = 0$
u_1
$\mu = \mu_1 \quad \sigma < 0$

Fig. 3

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